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# Interpolatory quadrature formulae with Chebyshev abscissae

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**Abstract**

We review interpolatory quadrature formulae, relative to the Legendre weight function on $[-1,1]$, having as nodes the zeros of any one of the four Chebyshev polynomials of degree $n$ and possibly one or both of the endpoints of the interval of integration. Some of the results we present here are new, and appear in the literature for the first time.


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*Keywords:* Interpolatory quadrature formulae; Chebyshev abscissae

## 1. Introduction

A natural way to approximate the integral $\int_{-1}^{1} f(t)\,\mathrm{d}t$ is to consider the $n$ distinct points $\tau_1, \tau_2, \ldots, \tau_n$, ordered decreasingly, in $(-1,1)$, find the interpolating polynomial $p_{n-1}(f;\tau_1,\tau_2,\ldots,\tau_n;t)$, and then write

$$\int_{-1}^{1} f(t)\,\mathrm{d}t \approx \int_{-1}^{1} p_{n-1}(f;\tau_1,\tau_2,\ldots,\tau_n;t)\,\mathrm{d}t.$$

That way we are led to a quadrature formula of the form

$$\int_{-1}^{1} f(t)\,\mathrm{d}t = \sum_{v=1}^{n} w_v f(\tau_v) + R_n(f). \tag{1.1}$$

By definition, (1.1) has degree of exactness $d$ at least $n-1$, i.e., $R_n(f)=0$ for all $f \in \mathbb{P}_{n-1}$, and is called an interpolatory quadrature formula relative to the Legendre weight function $w(t)=1$ on the interval $[-1,1]$.

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For $f \in C^{d+1}[-1,1]$, the error term of (1.1) can be expressed, by means of Peano's Theorem, in the form

$$R_n(f) = \int_{-1}^{1} K_d(t) f^{(d+1)}(t)\,\mathrm{d}t, \tag{1.2}$$

where $K_d$ is the $d$th Peano kernel for $R_n$. From (1.2), we immediately derive the estimate

$$|R_n(f)| \leqslant c_{d+1} \max_{-1\leqslant t\leqslant 1} |f^{(d+1)}(t)|, \quad c_{d+1} = \int_{-1}^{1} |K_d(t)|\,\mathrm{d}t. \tag{1.3}$$

Moreover, if $K_d$ does not change sign on $[-1,1]$, formula (1.1) is called definite, in particular, positive definite if $K_d \geqslant 0$ and negative definite if $K_d \leqslant 0$. In this case, (1.2) gives, by means of the Mean Value Theorem for integrals,

$$R_n(f) = \bar{c}_{d+1} f^{(d+1)}(\xi), \quad \bar{c}_{d+1} = \int_{-1}^{1} K_d(t)\,\mathrm{d}t, \quad -1 < \xi < 1 \tag{1.4}$$

(see, e.g., [7, Sections 2.5 and 4.3]).

If to the set of nodes in (1.1) we add either both or just one of the endpoints of the interval of integration, we end up with a different kind of an interpolatory formula. In the first case, we get

$$\int_{-1}^{1} f(t)\,\mathrm{d}t = w_0^* f(1) + \sum_{v=1}^{n} w_v^* f(\tau_v) + w_{n+1}^* f(-1) + R_n^*(f), \tag{1.5}$$

while in the second, we have

$$\int_{-1}^{1} f(t)\,\mathrm{d}t = w_0^{(+)} f(1) + \sum_{v=1}^{n} w_v^{(+)} f(\tau_v) + R_n^{(+)}(f), \tag{1.6}$$

or

$$\int_{-1}^{1} f(t)\,\mathrm{d}t = \sum_{v=1}^{n} w_v^{(-)} f(\tau_v) + w_{n+1}^{(-)} f(-1) + R_n^{(-)}(f). \tag{1.7}$$

Formulae (1.5) and (1.6)–(1.7) have degree of exactness at least $n+1$ and $n$, respectively, while for their error terms hold estimates analogous to (1.2)–(1.4).

Of particular interest are interpolatory formulae whose nodes and weights are explicitly known, not only because these formulae can be easily computed, but also since a lot of theoretical questions regarding them can be answered in a much simpler way. Up to now, the only known formulae of this kind are those having as nodes the zeros of any one of the four Chebyshev polynomials. These formulae have been extensively studied by several authors during the past sixty-five years, and this gave rise to a number of important properties for them such as the positivity of the weights, the precise degree of exactness, asymptotically optimal error bounds, the definiteness or nondefiniteness and the convergence for Riemann integrable functions on $[-1,1]$ as well as for functions with endpoint and/or interior singularities. In the following, we give a detailed overview of all these results, some of which are new and presented for the first time.

## 2. The quadrature formulae

We consider formulae of type (1.1) and (1.5)–(1.7) with the $\tau_v$ being zeros of any one of the four Chebyshev polynomials $T_n$, $U_n$, $V_n$ and $W_n$ of the first, second, third and fourth kind, respectively. These polynomials can be represented by the well-known trigonometric formulae

$$T_n(\cos\theta) = \cos n\theta, \quad U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}, \tag{2.1}$$

$$V_n(\cos\theta) = \frac{\cos(n+1/2)\theta}{\cos(\theta/2)}, \quad W_n(\cos\theta) = \frac{\sin(n+1/2)\theta}{\sin(\theta/2)}. \tag{2.2}$$

An immediate consequence of (2.1) and (2.2) is that the zeros of $T_n$, $U_n$, $V_n$ and $W_n$ can be computed explicitly, that is,

$$\tau_v^{(1)} = \cos\theta_v^{(1)}, \quad \theta_v^{(1)} = \frac{2v-1}{2n}\pi, \quad v = 1,2,\ldots,n, \tag{2.3}$$

$$\tau_v^{(2)} = \cos\theta_v^{(2)}, \quad \theta_v^{(2)} = \frac{v}{n+1}\pi, \quad v = 1,2,\ldots,n, \tag{2.4}$$

$$\tau_v^{(3)} = \cos\theta_v^{(3)}, \quad \theta_v^{(3)} = \frac{2v-1}{2n+1}\pi, \quad v = 1,2,\ldots,n, \tag{2.5}$$

$$\tau_v^{(4)} = \cos\theta_v^{(4)}, \quad \theta_v^{(4)} = \frac{2v}{2n+1}\pi, \quad v = 1,2,\ldots,n. \tag{2.6}$$

The study of formulae (1.1) and (1.5)–(1.7) with abscissae (2.3)–(2.6) started with Fejér in 1933 (cf. [8]). Next, we present an overview of the development in this area during the past sixty-five years.

### 2.1. Explicit formulae for the weights and positivity of them

Probably the most important property of the quadrature formulae in consideration is that the weights can be represented explicitly.

In what follows, $[\,\cdot\,]$ denotes the integer part of a real number. First, for the weights of formula (1.1), we have, when $\tau_v = \tau_v^{(1)}$,

$$w_v^{(1)} = \frac{2}{n}\left\{1 - 2\sum_{k=1}^{[n/2]} \frac{\cos 2k\theta_v^{(1)}}{4k^2-1}\right\}, \quad v = 1,2,\ldots,n; \tag{2.7}$$

and when $\tau_v = \tau_v^{(i)}$, $i = 2,3,4$,

$$w_v^{(i)} = \frac{2}{n+\alpha}\left\{1 - 2\sum_{k=1}^{[(n-1)/2]} \frac{\cos 2k\theta_v^{(i)}}{4k^2-1} - \frac{\cos 2[(n+1)/2]\theta_v^{(i)}}{2[(n+1)/2]-1}\right\}, \quad v = 1,2,\ldots,n, \; i = 2,3,4, \tag{2.8}$$

or, alternatively,

$$w_v^{(i)} = \frac{4\sin\theta_v^{(i)}}{n+\alpha}\sum_{k=1}^{[(n+1)/2]}\frac{\sin(2k-1)\theta_v^{(i)}}{2k-1}, \quad v=1,2,\ldots,n,\ i=2,3,4, \tag{2.9}$$

where

$$\alpha = \begin{cases} 1 & \text{if } i=2, \\ 1/2 & \text{if } i=3,4 \end{cases} \tag{2.10}$$

(see [7, pp. 84–85], [8, Sections 1 and 2] and [14, Theorem 2.1(a)]).

Similarly, for formula (1.5), one obtains, when $\tau_v = \tau_v^{(1)}$,

$$w_v^{*(1)} = \begin{cases} \dfrac{2}{n}\left\{1 - 2\displaystyle\sum_{k=1}^{n/2}\frac{\cos 2k\theta_v^{(1)}}{4k^2-1} + \frac{(-1)^{v-1}\cot\theta_v^{(1)}}{n^2-1}\right\} & \text{if } n \text{ is even,} \\ \dfrac{2}{n}\left\{1 - 2\displaystyle\sum_{k=1}^{(n-1)/2}\frac{\cos 2k\theta_v^{(1)}}{4k^2-1} + \frac{(-1)^{v-1}\csc\theta_v^{(1)}}{n^2-4}\right\} & \text{if } n \text{ is odd;} \end{cases} \quad v=1,2,\ldots,n, \tag{2.11}$$

$$w_0^{*(1)} = w_{n+1}^{*(1)} = \begin{cases} -\dfrac{1}{n^2-1} & \text{if } n \text{ is even,} \\ -\dfrac{1}{n^2-4} & \text{if } n \text{ is odd;} \end{cases} \tag{2.12}$$

when $\tau_v = \tau_v^{(2)}$,

$$w_v^{*(2)} = \frac{2}{n+1}\left\{1 - 2\sideset{}{^*}\sum_{k=1}^{[(n+1)/2]}\frac{\cos 2k\theta_v^{(2)}}{4k^2-1}\right\}, \quad v=1,2,\ldots,n, \tag{2.13}$$

$$w_0^{*(2)} = w_{n+1}^{*(2)} = \begin{cases} \dfrac{1}{(n+1)^2} & \text{if } n \text{ is even,} \\ \dfrac{1}{n(n+2)} & \text{if } n \text{ is odd,} \end{cases} \tag{2.14}$$

where the notation $\sum^*$ means that the last term in the sum should be halved if $n$ is odd; and when $\tau_v = \tau_v^{(i)}$, $i=3,4$,

$$w_v^{*(i)} = \frac{2}{n+1/2}\left\{1 - 2\sum_{k=1}^{[(n+1)/2]}\frac{\cos 2k\theta_v^{(i)}}{4k^2-1} + \frac{\sin(2[(n+1)/2]+1)\theta_v^{(i)}}{(4[(n+1)/2]^2-1)\sin\theta_v^{(i)}}\right\},$$
$$v=1,2,\ldots,n,\ i=3,4, \tag{2.15}$$

$$w_0^{*(3)} = w_{n+1}^{*(4)} = \begin{cases} -\dfrac{1}{n^2-1} & \text{if } n \text{ is even,} \\ -\dfrac{1}{n(n+2)} & \text{if } n \text{ is odd,} \end{cases} \qquad w_{n+1}^{*(3)} = w_0^{*(4)} = \begin{cases} \dfrac{2n-1}{(n^2-1)(2n+1)} & \text{if } n \text{ is even,} \\ \dfrac{2n+3}{n(n+2)(2n+1)} & \text{if } n \text{ is odd.} \end{cases} \tag{2.16}$$

Alternatively,

$$w_\nu^{*(i)} = \frac{8}{(n+\alpha)\sin\theta_\nu^{(i)}} \sum_{k=1}^{[(n+1)/2]} \frac{\sin(2k-1)\theta_\nu^{(i)}}{(3-2k)(4k^2-1)}, \quad \nu = 1,2,\ldots,n, \;\; i = 2,3,4, \tag{2.17}$$

with $\alpha$ given by (2.10) (see [2], [6], [7, p. 86] and [14, Theorem 3.1(a)]). Formulae (2.11), (2.12), (2.15), and (2.17) with $i=2$ are new. Their proof is sketched at the end of this section.

Finally, for the weights of formulae (1.6) and (1.7), we have, when $\tau_\nu = \tau_\nu^{(1)}$ and $n$ even,

$$w_\nu^{(+)(1)} = \frac{2}{n}\left\{1 - 2\sum_{k=1}^{n/2} \frac{\cos 2k\theta_\nu^{(1)}}{4k^2-1} + \frac{(-1)^{\nu-1}\cot(\theta_\nu^{(1)}/2)}{n^2-1}\right\}, \quad \nu = 1,2,\ldots,n, \tag{2.18}$$

$$w_0^{(+)(1)} = -\frac{2}{n^2-1}, \tag{2.19}$$

$$w_\nu^{(-)(1)} = \frac{2}{n}\left\{1 - 2\sum_{k=1}^{n/2} \frac{\cos 2k\theta_\nu^{(1)}}{4k^2-1} + \frac{(-1)^{\nu}\tan(\theta_\nu^{(1)}/2)}{n^2-1}\right\}, \quad \nu = 1,2,\ldots,n, \tag{2.20}$$

$$w_{n+1}^{(-)(1)} = -\frac{2}{n^2-1}, \tag{2.21}$$

while for $n$ odd, $w_\nu^{(+)(1)} = w_\nu^{(-)(1)} = w_\nu^{(1)}$, $\nu = 1,2,\ldots,n$ (cf. (2.7)), and $w_0^{(+)(1)} = w_{n+1}^{(-)(1)} = 0$; when $\tau_\nu = \tau_\nu^{(2)}$ and $n$ even,

$$w_\nu^{(+)(2)} = \frac{2}{n+1}\left\{1 - 2\sum_{k=1}^{n/2} \frac{\cos 2k\theta_\nu^{(2)}}{4k^2-1} + \frac{(-1)^{\nu}}{n+1}\right\}, \quad \nu = 1,2,\ldots,n, \tag{2.22}$$

$$w_0^{(+)(2)} = \frac{2}{(n+1)^2}, \tag{2.23}$$

$$w_\nu^{(-)(2)} = \frac{2}{n+1}\left\{1 - 2\sum_{k=1}^{n/2} \frac{\cos 2k\theta_\nu^{(2)}}{4k^2-1} + \frac{(-1)^{\nu-1}}{n+1}\right\}, \quad \nu = 1,2,\ldots,n, \tag{2.24}$$

$$w_{n+1}^{(-)(2)} = \frac{2}{(n+1)^2}, \tag{2.25}$$

while for $n$ odd, $w_\nu^{(+)(2)} = w_\nu^{(-)(2)} = w_\nu^{(2)}$, $\nu = 1,2,\ldots,n$ (cf. (2.8) or (2.9) with $i=2$), and $w_0^{(+)(2)} = w_{n+1}^{(-)(2)} = 0$; when $\tau_\nu = \tau_\nu^{(3)}$,

$$w_\nu^{(+)(3)} = \frac{2}{n+1/2}\left\{1 - 2\sum_{k=1}^{[n/2]} \frac{\cos 2k\theta_\nu^{(3)}}{4k^2-1} + \frac{(-1)^{n+\nu}\csc(\theta_\nu^{(3)}/2)}{2[n/2]+1}\right\}, \quad \nu = 1,2,\ldots,n, \tag{2.26}$$

$$w_0^{(+)(3)} = \frac{2(-1)^n}{2[n/2]+1}, \tag{2.27}$$

$$w_v^{(-)(3)} = \frac{2}{n+1/2}\left\{1 - 2\sum_{k=1}^{[n/2]} \frac{\cos 2k\theta_v^{(3)}}{4k^2-1}\right\}, \quad v = 1,2,\ldots,n, \tag{2.28}$$

$$w_{n+1}^{(-)(3)} = \frac{2}{(2[n/2]+1)(2n+1)}; \tag{2.29}$$

and when $\tau_v = \tau_v^{(4)}$,

$$w_v^{(+)(4)} = \frac{2}{n+1/2}\left\{1 - 2\sum_{k=1}^{[n/2]} \frac{\cos 2k\theta_v^{(4)}}{4k^2-1}\right\}, \quad v = 1,2,\ldots,n, \tag{2.30}$$

$$w_0^{(+)(4)} = \frac{2}{(2[n/2]+1)(2n+1)}, \tag{2.31}$$

$$w_v^{(-)(4)} = \frac{2}{n+1/2}\left\{1 - 2\sum_{k=1}^{[n/2]} \frac{\cos 2k\theta_v^{(4)}}{4k^2-1} + \frac{(-1)^{v-1}\sec(\theta_v^{(4)}/2)}{2[n/2]+1}\right\}, \quad v = 1,2,\ldots,n, \tag{2.32}$$

$$w_{n+1}^{(-)(4)} = \frac{2(-1)^n}{2[n/2]+1} \tag{2.33}$$

(see [15, Theorems 3.6–3.9(a)–(b)]).

Formulae (2.7)–(2.33) can be derived by applying the usual techniques in interpolatory quadrature (cf. [7, Section 2.5]) in conjunction with the Christoffel–Darboux identity (cf. [25, Section 3.2]) and (2.1)–(2.6). Especially (2.18)–(2.33) can be obtained in a much simpler way by means of certain formulae expressing the weights of (1.6) and (1.7) in terms of the corresponding weights in (1.1) (see [15, Proposition 2.1]). A similar treatment yields formulae (2.11)–(2.16). In this case, we use Proposition 2.1 below and proceed as in Theorems 3.6–3.9(a) in [15].

**Proposition 2.1.** *The weights of the interpolatory quadrature formula* (1.5) *are given by*

$$w_v^* = w_v + \frac{\int_{-1}^1 (t+\tau_v)p_n(t)\,\mathrm{d}t}{(\tau_v^2-1)p_n'(\tau_v)}, \quad v = 1,2,\ldots,n, \tag{2.34}$$

$$w_0^* = \frac{\int_{-1}^1 (1+t)p_n(t)\,\mathrm{d}t}{2p_n(1)}, \quad w_{n+1}^* = \frac{\int_{-1}^1 (1-t)p_n(t)\,\mathrm{d}t}{2p_n(-1)}, \tag{2.35}$$

*where* $p_n(t) = \prod_{v=1}^n (t-\tau_v)$ *and* $w_v$ *are the weights of the interpolatory formula* (1.1).

**Proof.** Setting $f(t) = (t^2-1)p_n(t)/(t-\tau_v)$ in formula (1.5), we have

$$w_v^* = \frac{1}{(\tau_v^2-1)p_n'(\tau_v)}\int_{-1}^1 \frac{(t^2-1)p_n(t)}{t-\tau_v}\,\mathrm{d}t$$

Table 1
Positivity of the weights for formulae (1.1) and (1.5)–(1.7) with nodes $\tau_\nu = \tau_\nu^{(i)}$, $i = 1,2,3,4$ (see Remark 1)

| | $\tau_\nu = \tau_\nu^{(1)}$ | $\tau_\nu = \tau_\nu^{(2)}$ | $\tau_\nu = \tau_\nu^{(3)}$ | $\tau_\nu = \tau_\nu^{(4)}$ |
|---|---|---|---|---|
| (1.1) | All<br>[8, Section 1] | All<br>[8, Section 2] | All<br>[24, Section 23] | All<br>[24, Section 23] |
| (1.5) | All but $w_0^{*(1)}$, $w_{n+1}^{*(1)}$, $n \geqslant 2$<br>[14, p. 85] | All<br>[10, p. 139] | All but $w_0^{*(3)}$<br>[14, Theorem 3.1(a)] | All but $w_{n+1}^{*(4)}$<br>[14, Theorem 3.1(a)] |
| (1.6) | All but $w_0^{(+)(1)}$, $n$ even<br>[15, Theorem 3.6(a)] | All<br>[15, Theorem 3.7(a)] | Some<br>[15, Theorem 3.8(a),(f)] | All<br>[15, Theorem 3.9(a)] |
| (1.7) | All but $w_{n+1}^{(-)(1)}$, $n$ even<br>[15, Theorem 3.6(b)] | All<br>[15, Theorem 3.7(b)] | All<br>[15, Theorem 3.8(b)] | Some<br>[15, Theorem 3.9(b)] |

$$= \frac{1}{(\tau_\nu^2 - 1)p_n'(\tau_\nu)} \int_{-1}^{1} \frac{(t^2 - \tau_\nu^2 + \tau_\nu^2 - 1)p_n(t)}{t - \tau_\nu}\,\mathrm{d}t$$

$$= \frac{1}{p_n'(\tau_\nu)} \int_{-1}^{1} \frac{p_n(t)}{t - \tau_\nu}\,\mathrm{d}t + \frac{1}{(\tau_\nu^2 - 1)p_n'(\tau_\nu)} \int_{-1}^{1} \frac{(t^2 - \tau_\nu^2)p_n(t)}{t - \tau_\nu}\,\mathrm{d}t = w_\nu + \frac{\int_{-1}^{1}(t + \tau_\nu)p_n(t)\,\mathrm{d}t}{(\tau_\nu^2 - 1)p_n'(\tau_\nu)}.$$

The formulae for $w_0^*$ and $w_{n+1}^*$ follow immediately if we set $f(t)=(t+1)p_n(t)$ and $f(t)=(t-1)p_n(t)$, respectively, in (1.5). $\square$

We now turn to the positivity of the weights. All results are summarized in Table 1. In each case we indicate which of the weights are positive and provide the appropriate reference. Whenever true, the positivity follows from the explicit formulae for the weights by virtue of trigonometric inequalities and results from the theory of orthogonal polynomials and the summation of series.

### 2.2. Degree of exactness

The precise degree of exactness for each of the quadrature formulae in consideration is given in Table 2 together with the appropriate references. In most cases, the degree of exactness is the one expected from the interpolatory nature of the quadrature formula (see the Introduction). An exception occurs in the so-called symmetric cases, i.e., formulae (1.1) and (1.5) with Chebyshev abscissae of the first or second kind, where, when $n$ is odd, the anticipated degree of exactness is increased by one.

### 2.3. Asymptotically optimal error bounds

In all quadrature formulae in consideration the best possible relation for $c_{d+1} = c_m$ in (1.3) as $m \to \infty$ is of the form $\mathrm{O}(2^{-m}m^{-k}(m!)^{-1})$. This can be concluded either from a lemma of Brass (cf. [3, Lemma]) or from a generalization of it due to Künz (cf. [11, Lemma "B"]). The optimal values of $k$ together with the appropriate references are tabulated in Table 3.

Table 2
Precise degree of exactness for formulae (1.1) and (1.5)–(1.7) with nodes $\tau_\nu = \tau_\nu^{(i)}$, $i = 1, 2, 3, 4$ (see Remark 1)

| | $\tau_\nu = \tau_\nu^{(1)}$ | $\tau_\nu = \tau_\nu^{(2)}$ | $\tau_\nu = \tau_\nu^{(3)}$ | $\tau_\nu = \tau_\nu^{(4)}$ |
|---|---|---|---|---|
| (1.1) | $n-1$ if $n$ is even<br>$n$ if $n$ is odd<br>[14, p. 85] | $n-1$ if $n$ is even<br>$n$ if $n$ is odd<br>[14, p. 85] | $n-1$<br><br>[14, Theorem 2.1(b)] | $n-1$<br><br>[14, Theorem 2.1(b)] |
| (1.5) | $n+1$ if $n$ is even<br>$n+2$ if $n$ is odd<br>[14, p. 85] | $n+1$ if $n$ is even<br>$n+2$ if $n$ is odd<br>[14, p. 85] | $n+1$<br><br>[14, Theorem 3.1(b)] | $n+1$<br><br>[14, Theorem 3.1(b)] |
| (1.6) | $n$<br>[15, Theorem 3.6(c)] | $n$<br>[15, Theorem 3.7(c)] | $n$<br>[15, Theorem 3.8(c)] | $n$<br>[15, Theorem 3.9(c)] |
| (1.7) | $n$<br>[15, Theorem 3.6(c)] | $n$<br>[15, Theorem 3.7(c)] | $n$<br>[15, Theorem 3.8(c)] | $n$<br>[15, Theorem 3.9(c)] |

Table 3
Optimal values of $k$ in the asymptotical error constants $c_m = O(2^{-m}m^{-k}(m!)^{-1})$ for formulae (1.1) and (1.5)–(1.7) with nodes $\tau_\nu = \tau_\nu^{(i)}$, $i = 1, 2, 3, 4$ (see Remark 1)

| | $\tau_\nu = \tau_\nu^{(1)}$ | $\tau_\nu = \tau_\nu^{(2)}$ | $\tau_\nu = \tau_\nu^{(3)}$ | $\tau_\nu = \tau_\nu^{(4)}$ |
|---|---|---|---|---|
| (1.1) | 2<br>[5, Theorem 3] | 1<br>[4, Theorem 2] for $n$ even<br>[13] for $n$ odd | 1<br>[12, Theorem 2<br>with $\alpha = -\beta = -1/2$] | 1<br>[12, Theorem 2<br>with $\alpha = -\beta = 1/2$] |
| (1.5) | 4<br>[5, Theorem 4] | 3<br>[5, Theorem 2] for $n$ even<br>[3, Proposition 1] for $n$ odd | 3<br>[12, Theorem 2<br>with $\alpha = -\beta = -1/2$] | 3<br>[12, Theorem 2<br>with $\alpha = -\beta = 1/2$] |
| (1.6) | 2<br>[15, Theorem 3.6(d)] | 1<br>[15, Theorem 3.7(d)] | 1<br>[15, Theorem 3.8(d)] | 2<br>[15, Theorem 3.9(d)] |
| (1.7) | 2<br>[15, Theorem 3.6(d)] | 1<br>[15, Theorem 3.7(d)] | 2<br>[15, Theorem 3.8(d)] | 1<br>[15, Theorem 3.9(d)] |

### 2.4. Definiteness or nondefiniteness

The question of definiteness or nondefiniteness has been settled for all quadrature formulae in consideration except for (1.6) and (1.7) with $\tau_\nu = \tau_\nu^{(3)}$ and $\tau_\nu = \tau_\nu^{(4)}$, respectively. The situation is depicted in Table 4, where "dp" stands for "positive definite", "dn" for "negative definite", "nd" for "nondefinite" and "ns" for "not settled". In the same table we provide the appropriate reference for each case. The definiteness is proved using one of the criteria developed by Steffensen in [21] and [22, pp. 155–157], Odgaard in [20], and Brass and Schmeisser in [4, Theorem 1] and [5, Corollary 2]. That last criterion is the most general and includes as special cases almost all of the previous

Table 4
Definiteness or nondefiniteness for formulae (1.1) and (1.5)–(1.7) with nodes $\tau_\nu = \tau_\nu^{(i)}$, $i = 1,2,3,4$ (see Remark 1)

| | $\tau_\nu = \tau_\nu^{(1)}$ | $\tau_\nu = \tau_\nu^{(2)}$ | $\tau_\nu = \tau_\nu^{(3)}$ | $\tau_\nu = \tau_\nu^{(4)}$ |
|---|---|---|---|---|
| (1.1) | nd if $n \geqslant 2$<br>dp if $n = 1$<br>[1, Proposition 2a, with $\lambda = 0$] | dp<br><br>[4, Theorem 2] for $n$ even<br>[13] for $n$ odd | nd<br><br>[14, Theorem 2.1(d)] | nd<br><br>[14, Theorem 2.1(d)] |
| (1.5) | nd if $n \geqslant 4$<br>dn if $n = 1$<br>dp if $n = 2,3$<br>[1, Proposition 2b, with $\lambda = 0$] | nd if $n \geqslant 2$<br>dn if $n = 1$<br><br>[1, Proposition 2b, with $\lambda = 1$] | nd if $n \geqslant 2$<br>dp if $n = 1$<br><br>[14, Theorem 3.1(d)] | nd if $n \geqslant 2$<br>dn if $n = 1$<br><br>[14, Theorem 3.1(d)] |
| (1.6) | nd if $n \geqslant 2$<br>dp if $n = 1$<br>[15, Theorem 3.6(e)] | dn if $n$ is even<br>dp if $n$ is odd<br>[15, Theorem 3.7(d)] | ns | nd<br><br>[15, Theorem 3.9(e)] |
| (1.7) | nd if $n \geqslant 2$<br>dp if $n = 1$<br>[15, Theorem 3.6(e)] | dp<br><br>[15, Theorem 3.7(d)] | nd<br><br>[15, Theorem 3.8(e)] | ns |

ones. The nondefiniteness on the other hand is shown in all cases using the criteria developed by Akrivis and Förster in [1, Proposition 1].

### 2.5. *Convergence for Riemann integrable functions and functions with endpoint and/or interior singularities*

Formulae (1.1) and (1.5)–(1.7) with $\tau_\nu = \tau_\nu^{(i)}$, $i = 1,2,3,4$, converge, i.e., the quadrature sum on the right tends to the integral on the left as $n \to \infty$, for all Riemann integrable functions on $[-1,1]$ (see [8], [14, pp. 85 and 94], [15, Theorems 3.6(f), 3.7(e), 3.8–3.9(f)] and [24, Section 23]). For those formulae that have positive weights (see Table 1), this is an immediate consequence of the well-known theorem of Steklov (cf. [23, pp. 176–179]) and Fejér (cf. [8, p. 291]). In case that some of the weights are negative the convergence follows either from a result of Rabinowitz (cf. [19, Lemma 1 with $w(t) = 1$]) or from some results of Steklov (cf. [23, pp. 174–176]) and Pólya (cf. [17, pp. 267–268]).

Furthermore, formulae (1.1), (1.6) and (1.7) also converge for functions having a monotonic singularity at 1 and/or $-1$. More specifically, let $M[-1,1)$ the class of functions $f$ which are continuous on the half-open interval $[-1,1)$, monotonic in some neighbourhood of 1, and such that $\lim_{x \to 1^-} \int_{-1}^{x} f(t)\,dt$ exists. The classes $M(-1,1]$ and $M(-1,1)$ are defined analogously, while $M$ stands for the union of the three classes. Now, formulae (1.1) with $\tau_\nu = \tau_\nu^{(i)}$, $i = 1,2,3,4$, converge for all $f \in M$. For the first two sets of nodes this was shown in [9, Section 3] while for the other two in [14, Theorem 4.1]. Also, formulae (1.6) and (1.7) with $\tau_\nu = \tau_\nu^{(i)}$, $i = 1,2,3,4$, converge for all $f \in M(-1,1]$ the first and for all $f \in M[-1,1)$ the second. This was demonstrated in [15, Theorem 4.1].

The convergence or nonconvergence of formulae (1.1) and (1.5) with $\tau_\nu = \tau_\nu^{(i)}$, $i = 1,2,3,4$, and $\tau_\nu = \tau_\nu^{(i)}$, $i = 1,2$, respectively, and in general with Jacobi abscissae, for functions with an interior singularity has been studied by Rabinowitz in [18, Theorem 6].

**Remark 1.** For $n$ odd, formulae (1.6) and (1.7) with $\tau_\nu = \tau_\nu^{(i)}$, $i = 1,2$, fall to the corresponding formula (1.1). The reader should keep that in mind while looking at the references provided in Tables 1–4.

**Remark 2.** Besides the quadrature formulae in consideration, also of particular interest are interpolatory formulae whose weights can be represented semiexplicitly, i.e., in terms of the unknown nodes. The only known formulae of this kind are those having as nodes the zeros of any one of the four Bernstein–Szegö polynomials. These polynomials can be represented as linear combinations of the corresponding Chebyshev polynomials (see [16]).

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